CANTOR'S CONTINUUM PROBLEM

Cristina Criste

Babeş-Bolyai University, Cluj-Napoca

Abstract: *One of the Georg Cantor's main intellectual preoccupations was the problem of the continuum. In 1878, this old philosophical problem, closely related to the concept of infinity, received a purely mathematical formulation, later known as the Continuum Hypothesis. In his attempt to demonstrate this conjecture, Cantor largely laid the foundations of set theory. I will firstly articulate the configuration of this mathematical expression and then I will present some of its philosophical aspects. The text will focus only on Cantor's work and not on set theory's later developments determined by the Zermelo-Fraenkel's axiomatization.*

Keywords: *Cantor, infinite, set theory, continuum hypothesis, philosophy.*

1. INTRODUCTION

There are certain notions that have situated themselves at the boundary between philosophy and mathematics, after these two fields of knowledge were gradually separated: namely, the concepts of infinite and infinitesimal, closely related to the fundamental ideas of analysis, to the concepts of number and magnitude, to those of continuous and discrete, limit, or the character of mathematical objects.

For a long time, starting with the Greeks, the idea of the continuum, strictly opposed to the idea of discrete, has been generally connected to the concepts of space, time and movement, i.e., to the intuition of the extended magnitude, although it is not easy to determine what exactly did the Greeks understand by magnitude.

These two fundamental characteristics of the traditional attitude towards the continuum – the opposition to the discrete and the intuitionistic approach to the extended magnitude – mean that a reconfiguration of the idea of the continuum would determine, among other possible things: a reconfiguration of its relationship with the discrete; a new idea of number; a rethinking of the problem of measure and magnitude; a new assessment of time, space and movement; a questioning of

An. Inst. de Ist. „G. Bariţiu" din Cluj-Napoca, Series Humanistica, tom. XIII, 2015, p. 177–193

basic epistemic premises, of what exactly infinite signifies, or what mathematics is, what is the nature of its objects, and what does it mean to do mathematics.

At the end of the nineteenth century, the set of real numbers came to be known as the arithmetical continuum, considered adequate to analytically represent all types of continuous phenomena. Cantor offered the necessary concepts, quite rigorously established, by which one is able to consider that points, intuitively discrete, can constitute a line, and that in general, a continuous domain of dimension n could be conceived of as a set of entities of dimension less than n. For Cantor, the old problems concerning the infinite, and implicitly those of the continuum, came to be reconsidered under the unifying language of sets. And the dialectic arithmetic-geometry received a certain mathematical closure: the arithmetical continuum is isomorphic with the linear geometrical continuum (situation known as the Cantor-Dedekind Axiom)[1].

These mathematical and semantic mutations meant, above all, acknowledging the idea of actual infinite, which assumes the idea of completing an infinite process. Basically, as it has been noticed many times, it is in principle "contradictory to speak of the completion of an unending process, but in practice many processes beg for completion, because they have a clear limit." (Stillwell, 2010, pp. 19–20). And in fact, this concept of limit is one which determined some of the most interesting changes in mathematics and, implicitly, in other areas of knowledge.

During the 19th century, many mathematicians tried to clarify different notions and to offer more solid foundations to their science, especially to analysis and, as a consequence, before the last quarter of the century, the world was already acquainted with the ε–δ definition of the limit, of continuity and, as such, to a more precise theory of the calculus.

2. THE FRAMEWORK

Interested himself in the foundation of analysis, Cantor started by studying the Fourier series and, in demonstrating the unicity of the representation of a function by a trigonometric series by gradually extending the conditions[2], he also began to profoundly reevaluate the distinctions to be made between the discrete and the continuous domains, introducing new relevant concepts.

Taking into consideration the research of Cauchy and Weierstrass regarding the sequences, the limits and the neighborhoods, he introduced his concept of real

[1] Provided the axioms of geometry are so selected to ensure this is the case.

[2] Especially (Cantor, 1872/1932). For the rest of the text, Cantor's articles will be cited in the same way: the first year represents the first publication of the article (in German or French), while the second, the year of publication of an English translation, or Zermelo's edition of almost all of Cantor's work (Zermelo: *Georg Cantor – Gesammelte Abhandlungen mathematischen und philosophischen Inhalts,* 1932), when English translations are not available, or considered unsatisfactory. The page will correspond to the second edition quoted. The names of the articles can be found, in the chronological order of appearance, in the general bibliography at the end of the text.

number: a real number is the limit *b* of a Cauchy sequence[3] of rational numbers $\{a_n\}$. On these new principles, the notion of number is regarded as carrying "in itself the germ for a necessary in itself and absolutely infinite extension"[4] (Cantor, 1872/1932, p. 95). And what his further research did show was how this extension would take two forms: the *Anzahl*, subsequently known as *ordinal number,* and the *power*, later developed as *cardinal number*, with the proviso that, in the case of finite numbers, the two are the same. Only Dedekind had offered a more rigorous definition of the real numbers[5]: namely, a 'cut', a 'gap' in the rational numbers (Dedekind, 1963, "Continuity and Irrational Numbers", § IV). In other words, a 'point' "is a whole universe to someone who knows that it is actually a cut in the infinite set of rational numbers." (Stillwell, 2010, p. 24).

If limit and, implicitly, the Bolzano-Weierstrass principle, constitute the mathematical basis in the creation of Cantor's set theory, the concept of derived set was his direct offspring of these building blocks: from this moment on and for some years, to Cantor, "the simplest and, at the same time, the most complete explanation for a continuum is based upon this idea [of derived sets]."[6] (Cantor, 1879/1932, p. 139). Given a set P, the derived set P' is the set obtained by taking all the limit points of the set P. This process of derivation can be iterated finitely and (later) infinitely many times, and the real numbers came to be characterized as the first derived set of the rational numbers.

Gradually, he became conscious of the fact that the infinite, in his actual aspect, is not a unitary concept, but one that admits different "degrees", a hierarchy which demands a special language. And in this new language, the articulation of the continuum's nature, identified with the real numbers, implies the proof of its uncountability.

3. THE UNCOUNTABILITY OF R AND THE POINT SETS

The article from 1874 (Cantor, 1874/2005) represents in its entirety a research in set theory. But what it directly demonstrates is the fact that the set of algebraic real numbers (viz. the roots of non-zero polynomials in one variable with integer coefficients) is countable. A set is finitely or infinitely countable if it can be indexed by the natural numbers, with a positive integer corresponding to each of its elements. What this idea implied was the existence of some sets that are not

[3] Fundamental sequence (*Fundamentalreihen*), in Cantor's words.

[4] "… in sich notwendigen und absolut unendlichen Erweiterung in sich" (Cantor, 1872/1932, p. 95).

[5] If we mainly consider the German world. The work of the French mathematician Méray on real numbers, published in 1869, was largely ignored by the German mathematicians.

[6] "Auf ihn wird, wie wir später zeigen wollen, die einfachste und zugleich vollständigste Erklärug resp. Bestimmung eines *Kontinuums* gegrüdet." (Cantor, 1879/1932, p. 139).

countable (the transcendental numbers, in this case), therefore of some infinite sets with different sizes.

In the following years, he maintained the generalizing tendency and introduced the concept of power[7] (*Mächtigkeit*, see Cantor [1878/1932]) as representing a one-to-one correspondence (a bijective function) between two sets of points. This concept would serve in the following years[8] as a means of distinguishing, in addition to the derivation process, between a continuous and a discontinuous distributed set on a *segment of a line*. And this last aspect is fundamental, because up until 1883, the study of the continuum concentrated only on the "linear point sets" (and afterwards on the concept of order).

Any set of natural numbers or the set of all natural numbers are discrete point-sets having a power of the first class, unlike the continuous set of real numbers, with a power of the second class.

Additionally, by demonstrating the somehow counterintuitive theorem that any continuous domain, whatever the number of dimensions, has the same power (equivalence relation) as any segment of the continuous line, like (0, 1), Cantor was convinced that the study of the continuous domains invites a focus only on the characteristics of the linear point-sets, viz. the study of the real number continuum (Cantor, 1878/1932, p. 121).

The focus on linear point-sets might reflect, as Dauben remarks, a reflection of Cantor's belief that $\mathbf{R}$ holds the greatest infinite order among point-sets (see Dauben, 1990, p. 59). But the theorem mentioned above suggests that he was also attempting to find point-sets of higher power than the real line. In fact, proving that any Euclidian interval of any dimension has the same power as the unit interval was apparently as surprising for him as for anybody[9].

And at the end of the study from 1878 he offered the now-famed Continuum Hypothesis, although, it appears, Cantor never used this formulation (see Moore, 1982, p. 41), but talked about the *Theorem of the two classes*[10]:

> [T]he question arises how the different parts of a continuous straight line, i.e. the different infinite manifolds of points that can be conceived in it, are related with respect to their powers. Let us divest this problem of its geometric guise, and understand (as has already been explained in §3) by a linear manifold of real numbers every conceivable totality [*Inbegriff*] of infinitely many, distinct real numbers. Then the question arises, into how many and which classes do the linear manifolds fall, if manifolds of the same power are placed into the same class, and manifolds of different power into different classes? By an inductive procedure, whose more exact presentation will not be given here, the theorem is suggested that the number of classes of linear

[7] Later called, after 1883, *cardinal number*.

[8] Starting with the Cantor's paper from 1879 (see Cantor, 1879/1932).

[9] "Je le vois, mais je ne le crois pas!", writes Cantor to Dedekind on 29th of June 1877 (Noether & Cavaillès, 1937, p. 34).

[10] The formulation appears in a letter of Cantor to Dedekind dated November 1882 (Von Noether, E. & Cavaillès, 1937, p. 876).

manifolds that this principle of sorting gives rise to is finite, and indeed, equals two. (Cantor, 1878/1932, p. 132; and in Ewald, 2005 II, p. 879).

The definition of the continuum was also discussed in three letters sent to Dedekind in 1882[11], from where one could deduce the intention of Cantor to offer a new definition, more general, of the continuum, which would also be applicable to continuous subsets of various dimensions, but connected in a continuous way. He later abandoned[12] this approach, but this direction represented the departing point in the creation of the descriptive set theory[13], which is pursuing the continuum hypothesis by investigating the simply described subsets of real numbers.

The abovementioned form of the continuum hypothesis represents what is now called the *Weak Continuum Hypothesis* and maintains that every infinite subset of **R** is either denumerable, or has the power of the continuum. The letter to Dedekind from November 1882 reprised it, but it also proposed a new theorem, the fact that every infinite subset pertaining to the second class is either numerable, or has the power of that class. It is likely, as Moore, too, remarked, that this formulation, together with the weak form of the hypothesis, had suggested to Cantor the strong version of the conjecture: **R** has the power of the second class[14] (see Moore, 1982, n. 12, pp. 41–2). The strong version of the hypothesis maintains that the real numbers can be well-ordered, and this last concept would be one to characterize a new stage in the development of set theory.

4. TRANSITIONS

The studies published after 1878, essentially motivated by the urge to prove the continuum hypothesis, were concentrated for the following five years on point-sets, while the notion of derived set still constituted the more general determination of a continuous domain.

Gradually, Cantor introduces new concepts[15], such as: dense-in-itself set (if and only if the elements of the set are also its limit points), isolated set (if and only if none of its limit points belongs to it), closed set (if and only if the set includes all its limit points), separated set (if and only if none of the subsets of a set is dense-in-itself).

[11] The correspondence Cantor-Dedekind in Ewald (2005): *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, II, pp. 872–878.

[12] This direct strategy was discussed as a theorem in the *Grundlagen,* §10 (Cantor, 1883/2005, pp. 903–906), but considered wrong in 1884 (Cantor, 1884/1932).

[13] Having as representatives, among others, Young, Hausdorff, the Moscovite or the Polish Schools of Mathematics.

[14] In the notation after 1895, $2^{\aleph_0} = \aleph_1$.

[15] In a collection of six articles from 1879 to 1884 about point sets, *Über unendlichelineare Punktmannigfaltigkeiten.*

Taking into account these distinctions, the continuum was initially considered a perfect set, namely closed (the continuous sets remain unaltered by the process of derivation)[16] and dense-in-itself[17]. In the light of the Cantor-Bendixson theorem[18], he showed that the continuum hypothesis is true for closed point-sets, although this is only a partial result of the continuum problem.

But because there are perfect sets which are nowhere-dense (the ternary set), Cantor also introduced the concept of connectivity[19], initially offered by Weierstrass in 1880 in terms of neighborhoods. In the circumstances, a continuous point-set is defined as a perfect connected set, *'perfect'* and *'connected'* representing "the necessary and sufficient properties of a point-continuum." (Cantor, 1883/2005, p. 906).

As Dauben remarks, connectivity represents the "equivalent form of everywhere-denseness", it accounts "for the metric property of the continuum" and frees "Cantor from having to introduce the auxiliary concept of the space with reference to which a given point-set could be said to be everywhere-dense" (Dauben, 1990, p. 110f.).

Generally speaking, and taking into consideration some of Ewald's remarks (2005, pp. 879–881), Cantor's attempt to prove the conjecture eventually took the form of two strategies. There is a direct one (also the starting point for the descriptive set theory), which uses the derived sets to measure the power (or cardinality) of a set by trying to show that the continuum hypothesis holds for sets whose first derived set is included in the initial set. And there is a second, indirect strategy, based on the ordinal theory of power, developed starting from 1883[20]. It is necessary though to emphasize the fact that the two strategies are not in contradiction, that there are aspects which could shed some light on the theory of sets as a whole, for example, the possible physical applications.

Trying to understand the essential differences between the continuous and the discrete, Cantor extended the conceptual characteristics of power to domains of multiple dimensions. And one very interesting result is the demonstration of the fact that continuous motion is possible in discontinuous spaces[21]. As a consequence, Cantor claims, it would be a mistake to hold on the traditionally assumed continuity of space, based upon an arbitrary evidence of continuous motion[22] (Cantor, 1882/1932, p. 157).

[16] $P^{(1)} \subset P$.

[17] *Grundlagen* §10 (Cantor, 1883/2005).

[18] Essentially, the Cantor-Bendixson theorem shows that the derivation process yields sets that are either denumerable, or perfect, or composed of both.

[19] A point set T of G_n (R^n) is connected if for any two points t and t' and any number ε arbitrarily small, there is a finite *Anzahl* of points $t_1, t_2, \ldots t_v$ of T, such as the distances tt_1, t_1t_2, t_2t_3, …, t_v t' are all less than ε (Cantor, 1883/2005, p. 906). It is one of the notions which inaugurates the topology of point sets. With this definition, the set of rational numbers is also connected, as Ferreirós, too, points out (see Ferreirós, 2007, p. 209).

[20] In the *Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen.*

[21] It represents the special case of a theorem (Cantor, 1882/1932, pp. 154–156).

[22] Cantor (1882/1932, p. 370)

5. APPROACHING THE UNCOUNTABILITY FROM THE SEQUENCE OF ORDINALS' PERSPECTIVE

What Ewald (2005) called the second strategy in proving the continuum hypothesis is based upon the introduction of transfinite numbers which could be used to 'count' the size of an infinite set, and thus place the power of the continuum in the hierarchy of transfinite ordinal numbers.

The underlying reason is the need to approach the continuous domains without assuming certain geometrical considerations, and which is felt even during the last two articles from the collection of six ones that tackles the infinite and linear point sets. Furthermore, one of them[23] discusses extensively different approaches to the continuum in the history of philosophy and theology[24]. The treatment of the transfinite numbers in 1883 is sometimes informal (if we consider the requirements of mathematical expression) and some of the terms rather imprecise, but the problems were resumed in the following years and received Cantor's final word in 1895 and 1897 in *Beiträge zur Begründung der transfiniten Mengenlehre* (see Cantor, 1895/2010, 1897/2010).

A premise for the new research tendency was the fact that the derivation of an initial point-set could be continued beyond a countably infinite number of iterations. On the other hand, the power of sets is not considered sufficient for distinguishing between the discrete and the continuum, or to distinguish different types of order for sets, so that the next stage in approaching the uncountability of the continuum was achieved through the general theory of order types.

The order type represents "the general concept under which all sets similar to the given set fall"[25] (in Grattan-Guinness, 1970, p. 87). In other words, it is the unique ordinal associated to a class of sets isomorphic among them (namely, there is a correspondence that maintains the order of the elements). The order types represent the common order of sets whose elements are considered, by the process of abstraction, "pure units", that is, ignoring the character of the elements. There are different types of order that could correspond to a set, which is a consequence of the defined order or succession[26]. Additionally, two sets simply ordered could be put in correspondence between them in various ways.

Until Cantor introduced the order types for the number classes, he did not possess the mathematical means to define in a very precise way any infinite cardinal

[23] Printed separately as well – hence its importance to Cantor – under the name *Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen* in 1883. (Cantor, 1883/2005).

[24] Apparently the reason why the fifth paper was not published in French.

[25] "…Allgemeinbegriff, unter welchen sämmtliche der gegebenen geordn." (Grattan-Guinness, 1970, p. 87).

[26] To Q, for example, one could attach the ordinal ω, when the rational numbers are arranged in the order of numerator-denominator sums, or its natural order, η.

which didn't belong to the first number class. This is the reason why the notation for the transfinite numbers is closely tied to the introduction of order types[27].

A first form of 'counting' infinite collections was the one-to-one correspondence, which is not exactly a definition of cardinality, but shows when two sets have the same cardinality. The creation of transfinite numbers meant the generalization or the continuation of the idea of counting to/into the infinite, expressed by the concept of *Anzahl*, later substituted by the concept of ordinal number: the types of infinite well-ordered sets can also be named infinite ordinals.

There are three principles of generating the ordinals: a repeated addition of units, taking limits and the one called *the principle of limitation*. Their employment makes possible the creation of number classes: the first, which contains all the sequences of natural numbers or the finite ordinals and denominated by the Greek small letter ω, and the second one, consisting of the elements ω, $\omega + 1$, $\omega\cdot 2$, $\omega\cdot n$, $\omega\cdot n + 8$, ω^{ω}…, viz., the countable infinite ordinals.

The limitation principle determines some 'breakings' in the sequence of transfinite numbers, making possible the 'jumps' into the second number class (or the continuum), individualizing and distinguishing it from the following classes.

On the other hand, the power or the cardinal number of a set M represents the "general concept that, by means of our active faculty of thought, arises from the aggregate M when we make abstraction of the nature of its various elements m and of the order in which they are given." (Cantor, 1895/2010, p. 86).

All the simply-ordered types corresponding to a certain cardinal α constitute a class of types [α], determined by the cardinal α. The cardinal β determined by the class [α] of types is different, because it belongs to the class taken in its entirety as a set of elements, each with cardinality α. In other words, the sets with the same type of order have the same cardinality.

A special significance is attached to those ordinals that cannot be obtained or 'reached' by an arithmetical operation (the ordinal arithmetic) of the elements that precede it, for example ω (or Ω). These initial ordinals play a fundamental role in describing a special function: the aleph.

Now all these definitions imply the idea of well-ordering, considered a law of thought[28] and necessary for establishing the series of transfinite numbers. A well-ordered set is a special kind of simply-ordered set. The last one, now called totally-ordered set, represents an order relation on a set such that for any two different elements a and b, either $a < b$, or $b < a$. To be considered well-ordered, an additional aspect must be fulfilled: every subset of the set must have a least element.

The well-ordered sets are essential to set theory for several reasons: characterizing the natural numbers, their types of order are fundamentally associated with the ordered sets of the first power; therefore they can be used as 'instruments'

[27] An aspect which was also noticed by Dauben (1990, p. 180).

[28] "…I shall discuss the law of thought that says that it is always possible to bring any well-defined set into the form of a well-ordered set – a law which seems to me fundamental and momentous and quite astonishing by reason of its general validity." (Cantor, 1883/2005, p. 886).

for creating transfinite numbers of higher classes[29]; they offer a unified account of the process of counting any set, including the set of real numbers; they provide valuable insights into Cantor's philosophical (and theological) assumptions.

But the first proof (and very controversial, for that matter) for the well-ordering theorem was offered only in 1904 by Zermelo. The 'problem' with the well-ordering was that it assumes the Axiom of Choice. There are many equivalent formulations for the Axiom of Choice, but it essentially maintains that for any set M there is a function f which associates to every subset S of the set M a unique element $f(S)$ which belongs to S. An element is thus "chosen" without a law to specify how to make the choice, and the things become more complicated in the case of an infinite set which would thus entail an infinity of arbitrary, simultaneous and independent choices. It is an axiom which stipulates only the existence of choice (the function), hence the reserves of those for whom the existence of a mathematical object is equivalent to its specification or definition.

We say today that by the Axiom of Choice, the cardinal numbers can be represented by the initial ordinals, namely the alephs: given a set M, its cardinal is considered to be the least ordinal among all the well-orderings of the set M. $\aleph_0$ thus represents the cardinality of the set of the ordinals initiated by ω0, and $\aleph_1$ the one initiated by ω1, viz. the well-ordered collection of numerable ordinals, each of cardinality $\aleph_0$, ordered by magnitude and shaping the least uncountable set (the ordinals of the form ω, $\omega + 1$, $\omega \cdot 2$, $\omega \cdot n$, $\omega \cdot n + 8$, ωω…,).

Accepting the Axiom of Choice, not only is the power of every set an aleph, but it becomes possible the cardinal comparability, which Cantor took it as a given[30], just like the well-ordering of every set. Cantor, like others of his contemporaries, assumed the possibility of infinite, arbitrary choices, without acknowledging the fact that they were using a new mathematical principle (see Moore, 1982).

The first explicit formulation of this axiom is offered by Zermelo in 1904 as well, when he proved the theorem of well-ordering[31]. Zermelo's results show that every cardinal number is an aleph and that there is no cardinal corresponding to a set which does not belong to the class of all cardinals.

Nevertheless, by introducing the order types and the theory of transfinite numbers in general, the continuum is 'counted' by the first uncountable ordinal and, in particular, constructed to come after $\aleph_0$ in the sequence of cardinals, by well-ordering and implicitly, by the Axiom of Choice. The idea was to solve the Continuum Hypothesis by showing where the continuum would be situated in the sequence of transfinite numbers, starting from the well-ordered sequence of natural numbers having cardinality $\aleph_0$.

[29] Although he acknowledged their existence, Cantor mainly focused on the first two classes.

[30] Only in 1915 does Hartogs prove the equivalence between the comparability of cardinals and the theorem of well-ordering.

[31] By well-ordering the subsets of the set to be well-ordered.

6. THE UNCOUNTABILITY OF THE CONTINUUM AND THE DIAGONAL ARGUMENT

To add a last piece in the creation of the image that would reflect Cantor's studies regarding the continuum problem, it remains to acknowledge the diagonal argument in demonstrating the uncountability of **R**, the other road to infinity, as some are calling it (see Stillwell, 2010). It was presented in 1891 (Cantor, 1891/2005) and has remained ever since a widely used instrument of proof in mathematics, logic, or computation[32].

Applying the diagonal argument to a set reveals the existence of an element that the set does not contain. By generalizing it, Cantor proved the theorem which bears his name, namely the fact that the set of all subsets of a set (what we now call the power set) and having the cardinality $2^{\aleph 0}$ cannot be put into a one-to-one correspondence with the initial set, being strictly greater than that. Cantor does not formulate the theorem in terms of *power set* (although he later used the concept, in his correspondence with Hilbert), but in terms of a certain collection of functions[33].

Unlike the 1874 demonstration of the uncountability of **R**, the diagonal argument applies to sets in general, a fact that entailed the reconsideration of the notion of set since, on one hand, it determines the emergence of sets with increasingly higher cardinality and, on the other hand, it is not at all obvious that these sets are given as already well-ordered by their construction.

However, not even this approach let Cantor to offer the solution to the continuum problem and, in fact, it is still not closed. In 1908, Zermelo offered the first axiomatisation of set theory and the research of Gödel in 1940 and Cohen's in 1963 proved that the conjecture of the continuum hypothesis is independent from the system of axioms Zermelo-Fraenkel with the Axiom of Choice, viz. the fact that it cannot be proved, nor disproved in this axiomatic system.

An important reason that has determined some mathematicians such as Baire or Cohen to reject the continuum hypothesis is the fact that there are two ways to produce an uncountable set[34]: taking all $\aleph_1$ ordinals of size $\aleph_0$, on one hand, and taking the $2^{\aleph 0}$ subsets of a set of size $\aleph_0$, on the other. But, they maintain, $\aleph_0$ and $2^{\aleph 0}$ are two very different things, because of the way in which they are produced: one by applying the first two principles of generation, the second by the power set procedure (with the aforementioned notes). In fact, Cohen considered that $2^{\aleph 0}$ would be larger than any aleph, being a very rich set obtained through an axiom

[32] An analogous form was employed by Gödel in his demonstration of the first incompleteness theorem, or by Turing regarding the *Entscheidungsproblem.*

[33] The collection of functions from a set L to a set with two elements and that has a cardinality strictly greater than L. In other words, he generalizes the concept of function, introducing arbitrary functions.

[34] Remark also found in Stillwell (2010, p. 40).

that would ignore a step by step construction of the continuum (see Cohen, 1966, p. 151). The continuum, it could be said, cannot be well-ordered and it wouldn't make any sense to talk about the well-ordering of the infinite sequences of the natural numbers[35].

7. PHILOSOPHICAL ASPECTS

From what was mentioned above, it becomes obvious the fact that the beginnings (or at least essential parts of them) of the abstract set theory could be found in the articulation and the settlement of the continuum problem. And there are arguments that maintain that the very act of laying down this problem is based on extra-mathematical motives[36], an aspect which Cantor himself was quite keen in acknowledging it. In a letter to Mittag-Leffler, for example, he exposes punctually what the problem he wants to solve is:

> I am also working at the applications of set theory to the natural theory of organisms, to which the existing principles of present mechanics cannot apply: this includes totally different principles and concepts from that of traditional mechanics. It is therefore necessary to create new mathematical instruments, which were essentially already created by me in parts of the set theory, as yet available and only some of them need further development. For fourteen years already I have been dealing with these ideas regarding the detailed exploration of everything organic, it constitutes the main reason for occupying myself with the tedious and little thanks-promising pursuit of point sets and of which, in this time, I have never lost sight.[37] (Meschkowski & Nilson, 1991, p. 202).

Cantor was always interested in discarding what he considered to be erroneous approaches to the continuum: in considering it an indivisible concept, an a priori intuition and situated beyond any attempt of analysis, especially a mathematical one. It was this kind of approaches that ultimately configured the idea of continuous functions and, as a consequence, a certain approach to the mathematical analysis that ignored the nature of the continuum based on the notion

[35] Additional information in Dauben (1990, pp. 240–270).

[36] See especially Ferreirós (2007).

[37] "Ausserdem bin ich mit Untersuchungen über Anwendungen der Mengenlehre auf die Naturlehre der Organismen beschaftiget, auf welche sich die bisherigen mechanischen Principien nicht anwenden lassen; es gehören dazu Grundlagen und Begriffe, welche von denen der bisherigen Mechanik toto genere verschieden sind. Dazu müssen auch ganz neue mathematicsche Hülfsmittel geschaffen werden, die aber im Wesentlichen in dem bisher von mir bearbeiteten Theile der Mengenlehre schon vorhanden und nur noch etwas weiter auszubilden sind. Mit diesen Ideen einer genaueren Ergründung des Wesens alles Organischen beschäftige ich mich schon seit 14 Jahren, sie bilden die eigentliche Veranlassung, werhalb ich das mühsame und wenig Dank verheissende Gaschäft der Untersuchung von Punctmengen unternommen und in diesem Zeitraum keinen Augenblick aus den Augen verloren habe." (Meschkowski &Nilson: *Georg Cantor: Briefe*, 1991, p. 202).

of the discrete, on the theory of real numbers. The cause for those undertakings, he believed, was the inaccurate assumptions of the so-called continuous magnitudes – time, space and movement.

Time was inextricably connected to the idea of infinite: when it was the case (depending on the theological or philosophical system), they were both found under the notion of an endless or unending process. This infinite is only “potentially” so, since we cannot grasp the elements of a set simultaneously, only one at a time. It could be assumed only as a process of producing the elements. Only in a ‘temporal’ setting could be said that every element comes at a definite ‘moment’.

It was the ‘discovery’ of uncountable sets that determined the declining role of time and motion in the mathematical thinking and therefore the acceptance (although not a unanimous one) of actual or completed infinities. To extend the order of natural numbers to infinity means to ask which is the next number, to add unit after unit. The answer offered by set theory is that infinity follows ***all*** the numbers.

Obviously, this result regarding the epistemology of the infinite would threaten an intuitive approach by which it would be possible to ‘attain’ the infinite only by assuming a generation and construction process that would never be completed, not even in the case of a countable set (a denumerable infinite set, like the set of natural numbers). But we cannot ignore the fact that, historically, this idea of possible infinite stood as the only acceptable type of infinity, mathematical or otherwise, consequence of the fact that each member can be found at a specific ‘moment’ in a sequence or by a process of counting.

Also, inwrought into the possible and temporal aspects of the infinite, is the idea of unboundedness. It essentially implies that infinity is the absence of limit. It cannot be said that Cantor changed the way limit was conceived in mathematics. Cauchy or Weierstrass are fundamental in this regard and Riemann was very keen in observing that unboundedness and infinite are two different types of relations[38] (see Riemann, 1866/1999, p. 160). Infinity does not mean the absence of limit. But integrated into set theory, the limits were essential in the creation of a new idea of number. Apart from being obvious in the definition of the irrational number, this aspect gains some interesting overtones in Cantor’s treatment of numbers in general. And a more philosophical example is his distinction between an *intrasubjective or immanent reality* of number and a *transsubjective or transient* one, but both coincident in the same number, consequence of the universe’s unity (Cantor, 1883/2005, p. 896).

All in all, the entire formalization (the shortcomings are rather negligible) introduced by Cantor (and Dedekind, too) for some old philosophical problems could be acknowledged as a detemporalized perspective in mathematics in general and its foundations in particular. As Stillwell emphasizes, the “mathematical existence is timeless” and the ideas of ‘process’ or ‘change’ are only figures of speech in this “timeless world” in which the natural numbers, if they exist, “they all exist”, and “the potential infinite is an actual infinite” (Stillwell, 2006, p. 201f.).

[38] Riemann, *“On the Hypotheses which lie at the Bases of Geometry”.*

The space represents, from a certain point of view, a different story. Not that its intuition would be prioritized, but the relation of order developed by Cantor starting from 1883 and eventually defining the transfinite world, and the continuum for that matter (with all the controversies), are not depleted of a certain 'spatial' content. They were fundamentally connected with some basic concepts, such as limit and point-set which, besides acquiring rapid acceptance among the mathematicians, were developed as topological concepts as well.

What is more, if we accept that "the essence of any 'space' in the most general sense" of the term is "not extension, but rather the ontological neutrality of all the positions defined by such a manifold" (Yourgrau, 1999, p. 10), Gödel's conception of sets as 'quasi-spatial' would make better sense[39].

Historically, the idea of continuity was essentially linked to that of extended magnitude, and although the Greeks never gave an explicit axiomatization for the theory of magnitudes (see Mancosu, 1996, p. 35), the exception represented by Aristotle has established as a fundamental ontological principle the fact that everything continuous, and thus infinitely divisible, cannot be constituted from that which is intuitively discrete and, in fact, the concept itself of continuity or that of infinite divisibility are inextricably linked to that of *dynamis*: for Aristotle, the infinite cannot be but potential.

The algebraization of geometry, started in the 17th century, definitely represents a decisive step in a process of mathematical unification. But there was, and still is, a fundamental drawback, an epistemic obstacle, consequence of our discursive framework of mind: we cannot grasp a whole, especially an infinite one. There occur then difficulties in accepting an actual infinite, or understanding the nature of our intuition. We tend to divide, to acknowledge the parts and then try to reconstruct the whole starting from these parts by a process of summation, unable to capture a complete net of relations, a unified element of our thought. One of the consequences is that we focus on the nature of the parts, be that a point, or even a line (in the case of a plane).

But when a point on the line is considered a limit, we avoid the confusion determined by trying to find out if a point is or is not a part of the continuum. We withstand the idea of trying to find out if the continuum is formed by divisible or indivisible elements: we could say that the continuum represents a system of parts without indivisible elements[40], but in fact, the distinction divisible-indivisible wouldn't be necessary anymore. Therefore, the number as limit comes as a solution to some old difficulties concerning the nature of continuity.

Considering some of Gödel's remarks regarding the space intuition, we begin by acknowledging that "[I]n space intuition, a point is not a part of the continuum but a limit between two parts" with all the points not adding to the line but only

[39] "Sets are quasi-spatial. They have an analogy to one and many, as well as to a whole and its parts." (Wang, 1996, 8.2.4, p. 254).

[40] A notion belonging to Gödel (see Wang, 1996, 7.3.19, p. 236).

making up “a scaffold (*Gerüsst*) or a collection of points of view”, and in the process acknowledging the fact that we cannot “longer adhere to the intuitive concept of points as limits but rather work with sums of limits as parts of the continuum” (Gödel in Wang, 1996, 7.3.19, p. 236).

Regardless of further philosophical developments, these distinctions maintain a relational status for the number. It is, in fact, a reflection of other relations: that between the infinite and the finite, or even that between a God and the world. In this new approach, the number as limit indicates the existence of other epistemic routes and options, in other words, other methods for accessing the infinite.

This is all set theory as a response to the continuum problem is all about: creating new methods of research, analysis and access to the infinite. By introducing the actual infinite, Cantor assumed the unraveling of the old tie between time and infinity: applying the diagonal argument and taking the power set, for example, this infinity becomes extensible, but not in the sense of a process that continues to the infinite by adding a unit or a finite quantity. Set theory, in its non-temporal essence, cannot be but of complete sets.

One of the consequences of set theory is the fact that we don’t have to question anymore the distinction between complete and incomplete multitudes. Another, and in the same context, is the fact that there is a break between time and infinity, and a reconsideration of the possible relation (if any) between the finite and the infinite.

A mathematical consequence of the new distinction between the finite and the infinite represented one of the pathways along which an infinity of arbitrary choices entered mathematics[41] (see Moore, 1982, p. 21), in other words, the Axiom of Choice. And eventually, the Axiom of Choice determined a transformation in what respects the proof regarding the existence of a mathematical object, including the option of a direct or constructivist proof.

Considering all the previous distinctions and Cantor’s statements, the notion of continuity implies, in an essential, way the concept of time, even though it (the time) represents nothing but an auxiliary concept that assures the relation (Cantor, 1883/2005, p. 904), and by no means something objective and absolute. Space, likewise, presumes an already conceptually completed continuum, and Cantor’s proof from 1882 that movement can be continuous even in a discontinuous space would be a fundamental proof in this respect. The assumption that space and time are a priori forms of intuition, he says, would contribute nothing to the knowledge of the continuum.

In the framework shaped by the discussions on the continuum, the concept of number, detemporalized and in its essence determined by the notions of limit and relation, is raised to another level. The concept of limit makes possible the integration of all kinds of numbers into the form of set theory. In the circumstances,

[41] Another, as the same author emphasizes, was the fact that the construction techniques of a mathematical object were not precisely delimited. (see Moore, 1996, p. 12).

the opposition generally attributed to the relationship between the finite and the infinite makes way for an analysis whose objects would be the description of this relation and all the changes it imposes in the configuration of a new epistemology. This means that the problem of the continuum essentially focuses on the properties of the finite-infinite 'relationship'. Or, in other words, the continuum problem "seems to lie at the very limit of the human capacity to understand the infinite." (Stillwell, 2010, p. 40).

A free simultaneous choice among infinite aspects and sets, independent of each other, but each with its equal claim to mathematical existence (or otherwise), might reveal interesting facts about those aspects and sets, about the whole that comprises them (and which is not the sum of its parts), and even, or especially, about the epistemic conditions an observer might possess. Acknowledging the actual infinite means assuming a different pattern of discovering tools and methods.

The consequence is Cantor's reconfiguration of the continuum problem into a new framework of mathematical concepts and, subsequently, new claims regarding the nature of reality itself.

8. CONCLUSIONS

Moving away from the point-sets, starting with 1883, Cantor emphasizes the ordinal character of the continuous domain, in opposition to the geometrical one, an aspect that could also highlight the fact that the concept of power alone is insufficient for making distinctions between the continuum and the discrete (see Dauben, 1990, p. 183) and for distinguishing their essence. And I am talking about essence because Cantor was very interested in defining the sets as units[42]: all the elements of set theory have the same ontological status as thoughts or images of the mind, in no way determined by spatial or temporal elements.

The different conceptual delimitations and the creation of new instruments for proof have shed new lights on the very notion of infinity, especially in its completed or actual aspect. Rather than a simple idea or process, the infinite often becomes a methodological concept, a type of general rule in which other forms of mathematical determination, like geometry, or arithmetic itself, are only some of its possible media of manifestation. The mathematicians could now be dealing with some incomprehensibly large sets as elements of the large cardinal theory, an extension of Zermelo-Fraenkel axiomatic system together with the Axiom of Choice and various axioms of infinity, a theory used to obtain results regarding arithmetic itself.

[42] "By an «aggregate» (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects *m* of our intuition or our thought. These objects are called the "elements" of M." (Cantor, 1895/2010, p.85).

In this new context, it is not really important whether the continuum hypothesis is true or not: its formulation meant the creation of a new language that has not ceased to develop ever since and to pose mathematics greater and greater challenges that shaped and enriched it, continuing to do so in the most unexpected ways. The wonder existed, but then came the moment when it had to be used in order to exit from a very old labyrinth.

REFERENCES

Cantor, G. (1932). Über die Ausdehnungeines Satzes aus der Theorie der trigonometrischen Reihen. In E. Zermelo, (Ed.), *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts* (pp. 92–102). Berlin: Springer. (Original work published 872).

Cantor, G. (1932). Ein Beitrag zur Mannigfaltigkeitslehre. In E. Zermelo, (Ed.), *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts* (pp. 119–33). Berlin: Springer. (Original work published 1878).

Cantor, G. (1932). Über unendliche, lineare Punktmannigfaltigkeiten, 1. In E. Zermelo, (Ed.), *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts* (pp. 139–45). Berlin: Springer. (Original work published 1879).

Cantor, G. (1932). Über unendliche, lineare Punktmannigfaltigkeiten, 3. In E. Zermelo, (Ed.), *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts* (pp. 149–57). Berlin: Springer. (Original work published 1882).

Cantor, G. (1932). Über unendliche, lineare Punktmannigfaltigkeiten, 6. In E. Zermelo, (Ed.), *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts* (pp. 210–46). Berlin: Springer. (Original work published 1884).

Cantor, G. (2005). Über eine Eigenschaft des Inbegriffes aller reellenalgebraischen Zahlen. In W. Ewald, (Ed. and Trans.), *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*. II. Oxford: Clarendon Press. (Original work published 1874).

Cantor, G. (2005). Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Einmathematisch-philosophischer Versuch in der Lehre des Unendlichen. In W. Ewald (Ed. and Trans.), *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*. II (pp. 878–920). Oxford: Clarendon Press. (Original work published in 1883).

Cantor, G. (2005). Über eine elementare Frage der Mannigfaltigkeitslehre. In Ewald (Ed. and Trans.). *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*. II (pp. 920–922). Oxford: Clarendon Press. (Original work published 1891).

Cantor, G. (2010). Beiträge zur Begründung der transfiniten Mengenlehre, 1. In *Contributions to the Founding of the Theory of Transfinite Numbers* (pp. 85–136). New York: Dover Publications. (Original work published 1895).

Cantor, G. (2010). Beiträge zur Begründung der transfiniten Mengenlehre, 2. In *Contributions to the Founding of the Theory of Transfinite Numbers* (pp. 137–201). New York: Dover Publications. (Original work published 1897).

Cohen, P. J. (1966). *Set Theory and the Continuum Hypothesis*. New York: W. A. Benjamin.

Dauben, J. W. (1990). *Georg Cantor: his mathematics and philosophy of the infinite*. Princeton University Press.

Dedekind, R. (1963). *Essays on the Theory of Numbers.* New York: Dover Publications.

Ewald, W. (Ed.) (2005). *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*. II. Oxford: Clarendon Press.

Ferreirós, J. (2007). *Labyrinth of Thought. A History of Set Theory and Its Role in Modern Mathematics*. Basel: Birkhäuser.

Grattan-Guinness, I. (1970). An unpublished paper by Georg Cantor: Principien einer Theorie der Ordnungstypen. Erste Mittheilung. *Acta Mathematica,* 124, pp. 65–107.

Mancosu, P. (1996). *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*. Oxford University Press.

Meschkowski, H. & Nilson, W. (Eds.). (1991). *Georg Cantor: Briefe*. Berlin: Springer.

Moore, G. H. (1982). *Zermelo's Axiom of Choice. Its Origins, Development, and Influence*. New York: Springer.

Riemann, B. (1999). On the Hypotheses which lie at The Foundations of Geometry. In M. A. Spivak (Trans.), *Comprehensive Introduction to Differential Geometry.* II. Houston: Publish or Perish. (Original work published in 1866).

Stillwell, J. (2006). *Yearning for the impossible. The surprising truths of mathematics.* Wellesley, Massachusetts: A K Peters.

Stillwell, J. (2010). *Roads to Infinity. The Mathematics of Truth and Proof.* Natick, Massachusetts: A.K. Peters.

Von Noether, E. & Cavaillès, J. (Eds.). (1937). *Briefwechsel Cantor-Dedekind.* Paris: Hermann.

Wang, H. (1996). *A Logical Journey: From Gödel to Philosophy.* Cambridge, Massachusetts: MIT Press.

Yourgrau, P. (1999). *Gödel meets Einstein: time travel in the Gödel universe.* Chicago: Open Court.